

SHARP WEIGHTED ESTIMATES FOR MULTI-LINEAR CALDERÓN-ZYGMUND OPERATORS ON NON-HOMOGENEOUS SPACES

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ABSTRACT. We establish sparse domination of multi-linear Calderón-Zygmund operators on non-homogeneous spaces. As a consequence we obtain sharp weighted bounds for multi-linear Calderón-Zygmund operators with respect to multi-linear $A_{\vec{p}}$ weights.

1. INTRODUCTION AND PRELIMINARIES

The theory of multi-linear singular integral operators has been developed extensively in past two decades. In 1970's Coifman and Meyer [1, 16] were one of the first to adopt a multi-linear point of view to study the commutators and paraproduct operators. In [4] Grafakos and Torres provided a systematic account of multi-linear Calderón-Zygmund operators. Recently, in [13] Lerner et al. developed a suitable notion of multi-linear weights and introduced an appropriate multi-linear maximal function. This maximal function plays the role of the classical Hardy-Littlewood maximal and helps in obtaining weighted estimates for the multi-linear Calderón-Zygmund operators. There have been many developments about weighted estimates for multi-linear operators in recent times. The most recent approach is the method of sparse domination of operators under consideration. This approach yields sharp weighted estimates for maximal operators and singular integral operators. In this paper, we extend the method of sparse domination to multi-linear Calderón-Zygmund operators defined on non-homogeneous spaces.

Recently in [15, 17] sparse domination technique have been introduced for Calderón-Zygmund operators in non-homogeneous setting using David-Mattila cells. In order to describe the results in details, we need to first recall the notion of upper doubling and geometrical doubling metric measure spaces.

Definition 1.1. [6, 15] We say that a metric measure space (X, d, μ) is *upper doubling* if there exist a dominating function $\lambda : X \times (0, \infty) \rightarrow (0, \infty)$ and a constant $C_\lambda > 0$ such that for every $x \in X$ the function $r \rightarrow \lambda(x, r)$ is non-decreasing and

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, r/2) \text{ for all } x \in X, r > 0.$$

Here $B(x, r)$ denotes the ball of radius r with center at $x \in X$.

Remark 1.1. In [8] authors showed that with the help of the dominating function λ , one can define a new domination function $\tilde{\lambda}$ satisfying an additional property

$$(1) \quad \tilde{\lambda}(x, r) \leq C_\lambda \tilde{\lambda}(y, r) \text{ for all } x, y \in X \text{ with } |x - y| \leq r$$

Therefore, without loss of any generality we may always assume that the dominating function λ satisfies the property (1).

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Definition 1.2. We say that a metric space (X, d) is *geometrically doubling* and has *doubling dimension* n if for given $0 < r \leq R$ and any ball $B \subset X$ of radius R , the r -separated subsets in B have the cardinality at most $C(\frac{R}{r})^n$.

We follow the notion of A_p weights presented in [19] and define multi-linear $A_{\vec{p}}(\mu)$ weights as follows.

Let $1 \leq p_1, \dots, p_m < \infty$ be such that $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$. Let \vec{P} denote the m -tuple $\vec{P} = (p_1, \dots, p_m)$. Given an m -tuple of weights $\vec{w} = (w_1, w_2, \dots, w_m)$, set

$$v_{\vec{w}} = \prod_{j=1}^m w_j^{p/p_j}.$$

Definition 1.3. Let $\rho \in [1, \infty)$. An m -tuple of weights $\vec{w} = (w_1, w_2, \dots, w_m)$ is said to belong to the multilinear $A_{\vec{P}}^\rho(\mu)$ -class if it satisfies the multilinear $A_{\vec{P}}^\rho(\mu)$ -condition

$$(2) \quad \sup_B \left(\frac{1}{\mu(\rho B)} \int_B v_{\vec{w}} d\mu \right)^{1/p} \prod_{j=1}^m \left(\frac{1}{\mu(\rho B)} \int_B w_j^{1-p'_j} d\mu \right)^{1/p'_j} \leq K < \infty.$$

where the supremum is being taken over all balls.

Here we follow the standard interpretation of the average $\left(\frac{1}{\mu(\rho B)} \int_B w^{1-p'} d\mu \right)^{\frac{1}{p'}}$ as $(\mu\text{-ess inf}_B w)^{-1}$ when $p_j = 1$.

1.1. Multi-linear Calderon-Zygmund operators.

Definition 1.4. An m -linear operator T which is bounded (weak-type) from $L^1(X) \times \dots \times L^1(X)$ into $L^{1/m, \infty}(X)$ and is represented for bounded functions f_j , $1 \leq j \leq m$, with bounded supports by

$$T(f_1, f_2, \dots, f_m)(x) = \int_{X^m} K(x, y_1, y_2, \dots, y_m) \prod_{j=1}^m f_j(y_j) d\mu(y_j)$$

for all $x \notin \bigcap_{j=1}^m \text{supp}(f_j)$, where the kernel K is locally integrable function defined away from the diagonal $x = y_1 = y_2 = \dots = y_m$ in X^{m+1} and satisfies

- Size condition:

$$(3) \quad |K(x, y_1, y_2, \dots, y_m)| \lesssim \min_{i=1,2,\dots,m} \frac{1}{\lambda(x, d(x, y_i))^m}$$

- Regularity conditions:

(4)

$$|K(x, y_1, y_2, \dots, y_m) - K(x', y_1, y_2, \dots, y_m)| \lesssim \min_{j=1,2,\dots,m} \frac{1}{\lambda(x, d(x, y_i))^m} \omega \left(\frac{d(x, x')}{\sum_{j=1}^m d(x, y_i)} \right)$$

whenever $d(x, x') \leq \frac{1}{2} \max_{j=1,2,\dots,m} d(x, y_i)$.

Also a similar regularity condition for each j ,

(5)

$$|K(x, \dots, y_j, \dots, y_m) - K(x, \dots, y'_j, \dots, y_m)| \lesssim \min_{j=1,2,\dots,m} \frac{1}{\lambda(x, d(x, y_i))^m} \omega \left(\frac{d(y_j, y'_j)}{\sum_{j=1}^m d(x, y_i)} \right)$$

whenever $d(y_j, y'_j) \leq \frac{1}{2} \max_{j=1,2,\dots,m} d(x, y_i)$.

where ω is a modulus of continuity satisfying the Dini condition.

For $r > 0$, we define the truncated operator T_r as,

$$T_r(f_1, f_2, \dots, f_m)(x) = \int_{\sum_{j=1}^m d(x, y_j)^2 > r^2} K(x, y_1, y_2, \dots, y_m) \prod_{j=1}^m f_j(y_j) d\mu(y_j).$$

The truncated maximal operator is given by

$$T^*(f_1, f_2, \dots, f_m)(x) = \sup_{r>0} |T_r(f_1, f_2, \dots, f_m)(x)|$$

1.2. David-Mattila cells. The notion of dyadic lattice in \mathbb{R}^n with a non-doubling measure μ was introduced by David-Mattila [3]. In [15], authors observed that the same construction of David-Mattila cells works in general in the case of a geometrically doubling metric measure space. The David-Mattila cells are the key to obtain the sparse domination of Calderón-Zygmund operators in [15]. In this paper we exploit their ideas and extend this to multi-linear Calderón-Zygmund operators defined on a geometrically doubling metric measure space (X, d, μ) . Let us first recall the notion of David-Mattila cells.

Lemma 1.2. [3, 15, 17] *Let (X, d, μ) be a geometrically doubling metric measure space with doubling dimension n and locally finite Borel measure μ . If W denotes the support of μ and $C_0 > 1$ and $A_0 > 5000C_0$ are two given constants, then for each integer k , there exists a partition of W into Borel sets $\mathcal{D}_k = \{Q\}_{Q \in \mathcal{D}_k}$ with the following properties*

- For each $k \in \mathbb{Z}$, the set W is disjoint union $W = \cup_{Q \in \mathcal{D}_k} Q$. Moreover, if $k < l$, $Q \in \mathcal{D}_l$, and $R \in \mathcal{D}_k$, then either $Q \cap R = \emptyset$ or $Q \subset R$.
- For each $k \in \mathbb{Z}$ and each cube $Q \in \mathcal{D}_k$, there exists a ball $B(Q) = B(z_Q, r(Q))$ with $z_Q \in W$ such that $A_0^{-k} \leq r(Q) \leq C_0 A_0^{-k}$ and $W \cap B(Q) \subset Q \subset W \cap 28B(Q) = W \subset B(z_Q, 28r(Q))$. Further, the collection $\{5B(Q)\}_{Q \in \mathcal{D}_k}$ is pairwise disjoint.
- Let \mathcal{D}_k^{db} denote the collection of cubes Q in \mathcal{D}_k satisfying doubling property

$$\mu(100B(Q)) \leq C_0 \mu(B(Q)).$$

- For the non-doubling cubes $Q \in \mathcal{D}_k \setminus \mathcal{D}_k^{db}$, we have that $r_Q = A_0^{-k}$ and

$$\mu(cB(Q)) \leq C_0^{-1} \mu(100cB(Q)) \text{ for all } 1 \leq c \leq C_0.$$

1.3. Multi-linear sparse operators. A family of measurable sets $\mathcal{S} = \{Q\}$ in X is said to be η -sparse, $0 < \eta < 1$, if for every $Q \in \mathcal{S}$ there exists a measurable set $E_Q \subset Q$ such that

- $\mu(E_Q) \geq \eta \mu(Q)$ and
- the sets $\{E_Q\}_{Q \in \mathcal{S}}$ are pairwise disjoint.

Given a sparse family \mathcal{S} in X and a large number $\alpha \geq 200$, a multi-linear version of the sparse operator is defined by

$$\mathcal{A}_{\mathcal{S}}(\vec{f})(x) = \sum_{Q \in \mathcal{S}} \left(\prod_{i=1}^m \frac{1}{\mu(\alpha B(Q))} \int_{30B(Q)} |f_i| d\mu \right) \chi_Q.$$

2. MAIN RESULTS

The main result of this paper is the following.

Theorem 2.1. *Let T be a multi-linear Calderón-Zygmund operator defined on an upper doubling, geometrically doubling metric measure space (X, d, μ) . Then for all exponents $1 < p_1, \dots, p_m < \infty$ satisfying $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$ and any multiple weight $\vec{w} \in A_{\vec{p}}(\mu)$, we have*

$$\|T(\vec{f})\|_{L^p(v_{\vec{w}})}^p \lesssim C_\omega \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}^p,$$

where

$$C_\omega = \begin{cases} \sup_Q \frac{v_{\vec{w}}(Q)^{p'_0} \prod_{i=1}^m \sigma_i(\alpha Q)^{\frac{pp'_0}{p_i}}}{\mu(\alpha B(Q))^{mp} \mu(Q)^{mp(p'_0-1)}} & \text{if } \frac{1}{m} < p \leq 1 \text{ and } p_0 = \min_i p_i \\ \sup_Q \frac{v_{\vec{w}}(Q) \prod_{i=1}^m \sigma_i(\alpha B(Q))}{\mu(\alpha B(Q))^{mp} v_{\vec{w}}(E_Q)^{\frac{1}{p'}} \prod_{i=1}^m \sigma_i(E_Q)^{\frac{1}{p_i}}} & \text{if } p \geq \max_i p'_i \\ \sup_Q \frac{v_{\vec{w}}(Q)^{p'_0} \prod_{i=1}^m \sigma_i(\alpha B(Q))^{p'_0}}{\mu(\alpha B(Q))^{mp'_0} v_{\vec{w}}(E_Q)^{\frac{p'}{p'}} \prod_{i=1}^m \sigma_i(E_Q)^{\frac{p'_0}{p_i}}} & \text{if } p'_0 = \max\{p'_1, \dots, p'_m\}. \end{cases}$$

We would like to remark that if the underlying measure is a doubling measure then in each of the above cases mentioned above, we get that $C_\omega^{\frac{1}{p}} \lesssim [\vec{w}]_{A_{\vec{p}}(\mu)}^{\max(1, \frac{p'_1}{p}, \dots, \frac{p'_m}{p})}$. Therefore, in view of [14], we obtain sharp constants for the case of doubling measures.

We shall follow the approach of Kranich and Volberg [15] and establish the above theorem using the method of sparse domination for multi-linear Calderón-Zygmund operators. To be precise, we shall show that

Theorem 2.2. *Let T be a multi-linear Calderón-Zygmund operator defined on an upper doubling, geometrically doubling metric measure space (X, d, μ) . If $\alpha \geq 200$ and $X' \subset X$ is a bounded set, then for integrable functions f_j , $1 \leq j \leq m$ with support contained in X' , we can find sparse families \mathcal{S}_k , $k \geq 0$, of David-Mattila cells such that the sparse domination*

$$(6) \quad T^*(\vec{f})(x) \leq \sum_{k=0}^{\infty} 100^{-k} \mathcal{A}_{\mathcal{S}_k}(\vec{f})(x)$$

holds pointwise μ -almost everywhere on X' with implicit constants independent of X' .

Next, we will obtain sharp weighted estimates for the multi-linear sparse operator $\mathcal{A}_{\mathcal{S}}$ and thereby obtain Theorem 2.1.

Proposition 2.3. *Let $1 < p_1, \dots, p_m < \infty$ be such that $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$. Then for any multiple weight $\vec{w} \in A_{\vec{p}}(\mu)$ we have*

$$\|\mathcal{A}_{\mathcal{S}}(\vec{f})\|_{L^p(v_{\vec{w}})}^p \lesssim C_\omega \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}^p,$$

where C_ω as in Theorem 2.1.

3. PROOFS OF RESULTS

3.1. Domination by sparse operators. In this section we establish the domination of multi-linear Calderón-Zygmund operators by sparse operators and thereby obtain a proof

of Theorem 2.2. For notational convenience, we shall write the proof in bilinear setting only.

Given a David-Mattila cell $Q_0 \in \mathcal{D}$, we shall denote by $\mathcal{D}(Q_0)$ the collection of cells Q contained in Q_0 .

Here we consider X^2 with the metric $\bar{d}((x_1, x_2), (y_1, y_2)) = \max_{i=1,2} d(x_i, y_i)$. Let $Q \in \mathcal{D}$ and $x \in Q$, define the truncated operator by

$$F(x, Q) = \int_{X^2 \setminus 30B(Q)^2} K(x, y_1, y_2) f_1(y_1) f_2(y_2),$$

where $B(Q)^2$ is a ball in (X^2, \bar{d}) which is $B(Q) \times B(Q)$. For a fixed cell Q_0 and $x \in Q_0$, the localized version of bilinear grand maximal truncated operator is defined by

$$\mathcal{M}_{T, Q_0}(f_1, f_2)(x) = \sup_{x \in P, P \in \mathcal{D}(Q_0)} \sup_{y \in P} |F(y, P)|.$$

The bilinear grand maximal truncated operator \mathcal{M}_{T, Q_0} is the key tool in obtaining the sparse domination. We would require to prove the end-point weak-type estimates for this operator. This is achieved by proving a pointwise relation between the operators \mathcal{M}_{T, Q_0} and T^* . In the process we need to study another bilinear maximal operator \mathcal{M}_λ , which may be thought of as an analogue of the classical Hardy-Littlewood maximal operator in bilinear setting. This is defined by

$$\mathcal{M}_\lambda(f_1, f_2)(x) = \sup_{r>0} \frac{1}{\lambda(x, r)^2} \int_{B(x, r)} |f_1(y_1)| d\mu(y_1) \int_{B(x, r)} |f_2(y_2)| d\mu(y_2).$$

It is easy to see that \mathcal{M}_λ is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2}, \infty}(\mu)$

Lemma 3.1. *For any cell $Q_0 \in \mathcal{D}$, we have*

$$|\mathcal{M}_{T, Q_0}(f_1, f_2)(x) - T^*(f_1, f_2)(x)| \leq C(\|\omega\|_{Dini} + C_K) \mathcal{M}_\lambda(f_1, f_2)(x)$$

for a.e $x \in Q_0$, with constant independent of Q_0 .

Proof. Let $Q \in \mathcal{D}$ be a cell and $x, x' \in Q$. Denote $J_k = (2^{k+1}30B(Q))^2 \setminus (2^k30B(Q))^2$ we have,

$$\begin{aligned} & |F(x, Q) - F(x', Q)| \\ & \leq \int_{X^2 \setminus 30B(Q)^2} |K(x, y_1, y_2) - K(x', y_1, y_2)| |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\ & \leq \sum_{k \geq 0} \int_{J_k} |K(x, y_1, y_2) - K(x', y_1, y_2)| |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\ & \leq \sum_{k \geq 0} \int_{J_k} \min_{i=1,2} \frac{1}{\lambda(x, d(x, y_i))^2} \omega \left(\frac{d(x, x')}{\sum_{j=1,2} d(x, y_j)} \right) |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\ & \lesssim_\lambda \sum_{k \geq 0} \int_{(2^{k+1}30B(Q))^2} \frac{1}{\lambda(x, 2^k r(Q))^2} \omega \left(\frac{56r(Q)}{2^k r(Q)} \right) |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\ & \lesssim_\lambda \|\omega\|_{Dini} \mathcal{M}_\lambda(f_1, f_2)(x). \end{aligned}$$

Denote

$$\tilde{T}_r(f_1, f_2)(x) = \int_{X^2 \setminus S_r(x)} K(x, y_1, y_2) f_1(y_1) f_2(y_2) d\mu(y_1) d\mu(y_2),$$

where $S_r(x) = \{\vec{y} \in X^2 : \max_{i=1,2} d(x, y_i) \leq r\}$. Let $x \in Q_0$ and fix a cell $P \in \mathcal{D}(Q_0)$ such that $x \in P$. Choose $r > 0$ to be such that $r \sim \text{diam}(P)$. Then,

$$\begin{aligned}
& |\tilde{T}_r(f_1, f_2)(x) - F(x, P)| \\
& \leq \int_{30B(P)^2 \setminus S_r(x)} |K(x, y_1, y_2)| |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\
& \leq \int_{30B(P)^2 \setminus S_r(x)} \min_{i=1,2} \frac{1}{\lambda(x, |x - y_i|)^2} |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\
& \leq C_K \int_{30B(P)^2 \setminus S_r(x)} \frac{1}{\lambda(x, 30r(P))^2} |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\
& \leq C_{K,\lambda} \frac{1}{\lambda(x, 30r(P))^2} \int_{30B(P)^2} |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\
& \leq C_{K,\lambda} \mathcal{M}_\lambda(f_1, f_2)(x).
\end{aligned}$$

Let $y \in P$ be any point, then by combining the above estimates, we get

$$(7) \quad |\tilde{T}_r(f_1, f_2)(x) - F(y, P)| \leq (C_{K,\lambda} + \|\omega\|_{Dini}) \mathcal{M}_\lambda(f_1, f_2)(x)$$

Thus from the Eq.(7) and the above estimate one can easily see that

$$\begin{aligned}
& |T_r(f_1, f_2)(x) - F(y, P)| \\
& \leq (\tilde{C}_{K,\lambda} + \|\omega\|_{Dini}) \mathcal{M}_\lambda(f_1, f_2)(x)
\end{aligned}$$

$$\sup_{y \in P} |F(y, P)| \leq T^*(f_1, f_2)(x) + (C_{K,\lambda} + \|\omega\|_{Dini}) \mathcal{M}_\lambda(f_1, f_2)(x)$$

Finally, taking the supremum over all cells P containing x , we obtain:

$$|\mathcal{M}_{T,Q_0}(f_1, f_2)(x) - T^*(f_1, f_2)(x)| \leq (C_{K,\lambda} + \|\omega\|_{Dini}) \mathcal{M}_\lambda(f_1, f_2)(x).$$

Following the arguments in [5, 18], T^* is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2},\infty}(\mu)$. Thus \mathcal{M}_{T,Q_0} is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2},\infty}(\mu)$. This completes the proof of Lemma 3.1. \square

The proof of the sparse domination (6) is constructive and follows a recursive argument. We shall prove a recursive formula involving the operator \mathcal{M}_{T,Q_0} . Before proceeding further, we set some notation. For $\alpha \geq 200$ and a cell Q , we write for bilinear averages

$$A(\vec{f}, Q) = \prod_{i=1}^2 \frac{1}{\mu(\alpha B(Q))} \int_{30B(Q)} |f_i| d\mu.$$

Also, for a given cell Q , we use the notation

$$\Theta(Q) = \frac{\mu(\alpha B(Q))^2}{\lambda(z_Q, r(Q))^2}$$

We shall prove the following recursive formula.

Lemma 3.2. *For any two nested cells $Q \subset \hat{Q}$, we have*

$$(8) \quad \mathcal{M}_{T,\hat{Q}}(f_1 1_{30B(\hat{Q})}, f_2 1_{30B(\hat{Q})})(x) \leq C \Theta(\hat{Q}) A(\vec{f}, \hat{Q}) + \mathcal{M}_{T,Q}(f_1 1_{30B(Q)}, f_2 1_{30B(Q)})(x)$$

We construct the David-Mattila cells such that $Q \subset \hat{Q}$ implies $30B(Q) \subset 30B(\hat{Q})$.

Proof. For every cell $Q \in \mathcal{D}$, $x \in Q$, and every cell $P \in \mathcal{D}(\hat{Q})$ with $x \in P$, we have

$$\begin{aligned} |F(y, P)| &= \left| \int_{30B(\hat{Q})^2 \setminus 30B(P)^2} K(y, z_1, z_2) f_1(z_1) f_2(z_2) d\mu(z_1) d\mu(z_2) \right| \\ &= \left| \int_{30B(Q)^2 \setminus 30B(P)^2} K(y, z_1, z_2) f_1(z_1) f_2(z_2) d\mu(z_1) d\mu(z_2) \right. \\ &\quad \left. + \int_{30B(\hat{Q})^2 \setminus 30B(Q)^2} K(y, z_1, z_2) f_1(z_1) f_2(z_2) d\mu(z_1) d\mu(z_2) \right| \\ &\lesssim \mathcal{M}_{T,Q}(f_1 1_{30B(Q)}, f_2 1_{30B(Q)})(x) + \frac{1}{\lambda(x, r(Q))^2} \int_{30B(\hat{Q})^2} |f_1(z_1)| |f_2(z_2)| d\mu(z_1) d\mu(z_2). \end{aligned}$$

Therefore, we get the recursive formula (8). \square

We know from [3, 15] that the quantities $\Theta(Q)$ are bounded by 1 and they decrease exponentially for a nested sequence of non-doubling cells. More precisely, we know that

Lemma 3.3. [3, 15] *Let l_0 be the maximal number with $100^{l_0} < \frac{C_0}{\alpha}$ and suppose that $C_0^{l_0/2} > C_\lambda^{\lceil \log_2 A \rceil}$, where C_λ is the doubling constant of the dominating function λ . Let*

$$Q_0 = \hat{Q}_1 \supset Q_1 = \hat{Q}_2 \supset Q_2 = \hat{Q}_3 \supset \dots$$

be a nested family of cubes such that Q_1, Q_2, \dots are non-doubling. Then,

$$\Theta(Q_k) \lesssim C_0^{-kl_0} \Theta(Q_0)$$

Lemma 3.4. *Let $Q_0 \in \mathcal{D}^{db}$ be a doubling cell. Then there exists a subset $\Omega \subset Q_0$, a collections of pairwise disjoint cubes $\mathcal{C}_n(Q_0) \subset \mathcal{D}$, $n = 1, 2, 3, \dots$ contained in Ω , and a collection of pairwise disjoint doubling cubes $\mathcal{F}(Q_0) \subset \mathcal{D}^{db}$ contained in Ω such that*

1. $\mu(\Omega) \leq \frac{1}{2} \mu(Q_0)$
2. For every $P \in \mathcal{F}$ and $Q \in \mathcal{C}_n(Q_0)$ either $P \subset Q$ or $P \cap Q = \emptyset$,
3. $\mathcal{M}_{T,Q_0}(\vec{f}) 1_{Q_0} \leq \sum_{P \in \mathcal{F}(Q_0)} \mathcal{M}_{T,P}(\vec{f}) 1_P + CA(\vec{f}, Q_0) + C \sum_{n=1}^{\infty} 100^{-n} \sum_{Q \in \mathcal{C}_n(Q_0)} A(\vec{f}, Q) 1_Q.$

Proof. The weak-type $(1, 1, 1/2)$ boundedness of the operator \mathcal{M}_{T,Q_0} implies that for large enough $M > 0$, the set

$$\Omega = \{x \in Q_0 : \mathcal{M}_{T,Q_0}(f_1, f_2)(x) > MA(\vec{f}, Q_0)\}$$

satisfies the desired property (1).

Since Ω can be decomposed into a collection of maximal and hence disjoint David-Mattila cells, say $\mathcal{C}_0(Q_0)$, we get

$$\sum_{Q \in \mathcal{C}_0(Q_0)} \mu(Q) \leq \frac{1}{2} \mu(Q_0).$$

For a cell $Q \in \mathcal{C}_0(Q_0)$ and $x \in Q$, the maximality of Q implies that

$$\sup_{y \in \tilde{Q}} |F(x, \tilde{Q})| \leq MA(\vec{f}, Q_0)$$

for any \tilde{Q} such that $Q \subset \tilde{Q} \subset Q_0$. Therefore, we have

$$\mathcal{M}_{T,Q_0}(f_1, f_2)(x) \leq MA(\vec{f}, Q_0) + \mathcal{M}_{T,\tilde{Q}}(f_1, f_2)(x).$$

The above estimate together with the recursive formula (8) yield

$$\mathcal{M}_{T,Q_0}(f_1, f_2)(x) \leq MA(\vec{f}, Q_0) + C\Theta(\hat{Q})A(\vec{f}, \hat{Q}) + \mathcal{M}_{T,Q}(f_1 1_{30B(Q)}, f_2 1_{30B(Q)})(x).$$

Next, we separate the cells with doubling property and set

$$\mathcal{F} = \{Q \in \mathcal{C}_0(Q_0) : Q \text{ is doubling}\} \text{ and } \mathcal{C}_1 = \mathcal{C}_0(Q_0) \setminus \mathcal{F}.$$

Now for a cell $Q \in \mathcal{D}$ such that $\hat{Q} \in \mathcal{C}_1$, if Q is doubling we will put it in the basket \mathcal{F} otherwise it goes to \mathcal{C}_2 . We will follow this process inductively. Now by Lemma 3.3 we know that the quantities $\Theta(Q)$, $Q \in \mathcal{C}_n$ decay exponentially thus summing over them, we get the desired result. \square

Recall Lemma 3.1 and note that in order to prove sparse domination of the operator T^* , it is sufficient to do so for the maximal operator \mathcal{M}_λ . The next lemma establishes this sparse domination. We require to consider the localized version $\mathcal{M}_\lambda^{d,Q_0}$ of \mathcal{M}_λ for doubling cell Q_0 , which is defined by taking the supremum over cells contained in $\mathcal{D}(Q_0)$, that is,

$$\mathcal{M}_\lambda^{d,Q_0}(\vec{f})(x) = \sup_{x \in P, P \in \mathcal{D}(Q_0)} \prod_{i=1}^2 \frac{1}{\lambda(z_P, r(P))} \int_{30B(P)} |f_i| d\mu.$$

Lemma 3.5. *For every doubling cell $Q_0 \in \mathcal{D}^{db}$ there exist a subset $\Omega \subset Q_0$, a collections of pairwise disjoint cubes $\mathcal{C}_n(Q_0) \subset \mathcal{D}$, $n = 1, \dots$ contained in Ω , and a collection of pairwise disjoint doubling cubes $\mathcal{F}(Q_0) \subset \mathcal{D}^{db}$ contained in Ω such that*

1. $\mu(\Omega) \leq \frac{1}{2}\mu(Q_0)$
2. For every $P \in \mathcal{F}$ and $Q \in \mathcal{C}_n(Q_0)$ then either $P \subset Q$ or $P \cap Q = \emptyset$,
3. $\mathcal{M}_\lambda^{d,Q_0}(\vec{f})\chi_{Q_0} \leq \sum_{P \in \mathcal{F}(Q_0)} \mathcal{M}_\lambda^{d,P}(\vec{f})\chi_P + CA(\vec{f}, Q_0) + C \sum_{n=1}^{\infty} 100^{-n} \sum_{Q \in \mathcal{C}_n(Q_0)} A(\vec{f}, Q)\chi_Q$

The proof of this lemma is similar to the previous lemma. Thus we skip the proof.

To have recursive application of Lemma 3.4 we will encounter cubes which are doubling as well as non-doubling. At this point we follow the arguments as given in [15] to find a sparse collection. This completes the proof of Theorem 2.2.

3.2. Sharp weighted inequalities for multi-linear oparse operators. We begin with the following observation about multiple weights. For details one can look into [14].

Lemma 3.6. *Let $1 < p_1, \dots, p_m < \infty$ be such that $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$. If $\vec{w} = (w_1, \dots, w_m) \in$*

$A_{\vec{P}}(\mu)$, then $\vec{w}^i = (w_1, \dots, w_{i-1}, v_{\vec{w}}^{1-p'}, w_{i+1}, \dots, w_m) \in A_{\vec{P}^i}$ with $[\vec{w}^i]_{A_{\vec{P}^i}} = [\vec{w}]_{A_{\vec{P}}}^{\frac{p_i'}{p}}$, where $\vec{P}^i = (p_1, \dots, p_{i-1}, p', p_{i+1}, \dots, p_m)$.

Proof of Proposition 2.3: We first consider the case $\frac{1}{m} < p \leq 1$. Let's assume $p_1 = \min_{i=1,2} \{p_1, p_2\}$. Now,

$$\begin{aligned} & \int_X \mathcal{A}_S(\vec{f}_\sigma)^p v_{\vec{w}} d\mu \\ & \leq \sum_{Q \in \mathcal{S}} \left(\prod_{i=1}^2 \frac{1}{\mu(\alpha B(Q))} \int_{30B(Q)} |f_i| \sigma_i \right)^p v_{\vec{w}}(Q) \\ & \leq \left(\sup_Q K_Q \right) \sum_{Q \in \mathcal{S}} \frac{\mu(B)^{mp(p'_1-1)}}{v_{\vec{w}}(Q)^{p'_1-1} \prod_{i=1}^2 \sigma_i(\alpha Q)^{\frac{pp'_1}{p_i}}} \left(\prod_{i=1}^2 \int_{30B(Q)} |f_i| \sigma_i \right)^p \end{aligned}$$

where

$$K_Q = \frac{v_{\vec{w}}(Q)^{p'_1} \prod_{i=1}^2 \sigma_i(\alpha Q)^{\frac{pp'_1}{p_i}}}{\mu(\alpha B(Q))^{mp} \mu(Q)^{mp(p'_1-1)}}$$

Now we observe that

$$\mu(Q)^{mp(p'_1-1)} \lesssim \mu(E(Q))^{mp(p'_1-1)} \leq v_{\vec{w}}(Q)^{p'_1-1} \prod_{i=1}^2 \sigma_i(E_Q)^{\frac{p(p'_1-1)}{p_i}}$$

and

$$v_{\vec{w}}(Q)^{p'_1-1} \leq v_{\vec{w}}(Q)^{p'_1-1} \quad \text{and} \quad \sigma_i(E_Q)^{\frac{p(p'_1-1)}{p_i} - \frac{p}{p_i}} \leq \sigma_i(Q)^{\frac{pp'_1}{p_i} - p}$$

Now using all the estimates above, we get

$$\begin{aligned} & \int_X \mathcal{A}_S(\vec{f}_\sigma)^p v_{\vec{w}} d\mu \\ & \leq \left(\sup_Q K_Q \right) \sum_{Q \in \mathcal{S}} \prod_{i=1}^2 \left(\frac{1}{\sigma_i(\alpha B(Q))} \int_{30B(Q)} |f_i| \sigma_i \right)^p \sigma_i(E_Q)^{\frac{p}{p_i}} \\ & \leq \left(\sup_Q K_Q \right) \prod_{i=1}^2 \left\{ \sum_{Q \in \mathcal{S}} \left(\frac{1}{\sigma_i(\alpha B(Q))} \int_{30B(Q)} |f_i| \sigma_i \right)^{p_i} \sigma_i(E_Q) \right\}^{\frac{p}{p_i}} \\ & \leq \left(\sup_Q K_Q \right) \prod_{i=1}^2 \left(\sum_{Q \in \mathcal{S}} \int_{E_Q} M_{\sigma_i d\mu}(f_i)^{p_i} \sigma_i d\mu \right)^{\frac{p}{p_i}} \\ & \leq \left(\sup_Q K_Q \right) \prod_{i=1}^2 \|M_{\sigma_i d\mu}(f_i)\|_{L^{p_i}(\sigma_i d\mu)}^p \\ & \leq \left(\sup_Q K_Q \right) \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i d\mu)}^p \end{aligned}$$

Write

$$C_\omega = \sup_Q K_Q = \sup_Q \frac{v_{\vec{w}}(Q)^{p'_1} \prod_{i=1}^2 \sigma_i(\alpha Q)^{\frac{pp'_1}{p_i}}}{\mu(\alpha B(Q))^{mp} \mu(Q)^{mp(p'_1-1)}}$$

If the underlying measure is doubling then $\mu(\alpha B(Q))$ and $\mu(Q)$ are comparable.

$$C_\omega = \sup_Q K_Q \leq \frac{v_{\vec{w}}(\alpha B(Q))^{p'_1} \prod_{i=1}^2 \sigma_i(\alpha B(Q))^{\frac{pp_1}{p'_i}}}{\mu(\alpha B(Q))^{mp} \mu(\alpha B(Q))^{mp(p'_1-1)}} \leq [\vec{w}]_{A_{\vec{P}}(\mu)}^{p'_1}$$

Therefore, for doubling measure, we have

$$\|\mathcal{A}_S(\vec{f})\|_{L^p(v_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}(\mu)}^{\frac{p'_1}{p}} \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)},$$

which is sharp for doubling measures.

Next, consider the case when $p \geq \max_i p'_i$.

It is sufficient to prove

$$\|\mathcal{A}_S(\vec{f}_\sigma)\|_{L^p(v_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}(\mu)} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}$$

for all f_j in a dense class. Since, $p > 1$, we use the duality to estimate the above. For a non-negative function $g \in L^{p'}(v_{\vec{w}})$ consider

$$\begin{aligned} \int_X \mathcal{A}_S(\vec{f}_\sigma) g v_{\vec{w}} d\mu &\leq \sum_{Q \in \mathcal{S}} \int_Q g v_{\vec{w}} d\mu \cdot \prod_{i=1}^m \frac{1}{\mu(\alpha B(Q))} \int_{30B(Q)} |f_i| \sigma_i d\mu \\ &= \sum_{Q \in \mathcal{S}} \frac{v_{\vec{w}}(Q) \prod_{i=1}^m \sigma_i(200B(Q))}{\mu(\alpha B(Q))^m} \frac{1}{v_{\vec{w}}(Q)} \int_Q g v_{\vec{w}} d\mu \prod_{i=1}^m \frac{1}{\sigma_i(200B(Q))} \int_{30B(Q)} |f_i| \sigma_i d\mu \\ &\lesssim \sup_Q \frac{v_{\vec{w}}(Q) \prod_{i=1}^m \sigma_i(200B(Q))}{\mu(\alpha B(Q))^m v_{\vec{w}}(E_Q)^{\frac{1}{p}} \prod_{i=1}^m \sigma_i(E_Q)^{\frac{1}{p_i}}} \left[\sum_{Q \in \mathcal{S}} \left(\frac{1}{v_{\vec{w}}(Q)} \int_Q g v_{\vec{w}}^{p'} v_{\vec{w}}(E_Q) \right)^{\frac{1}{p}} \right] \\ &\quad \prod_{i=1}^m \left[\sum_{Q \in \mathcal{S}} \left(\frac{1}{\sigma_i(200B(Q))} \int_{30B(Q)} |f_i| \sigma_i^{p_i} \sigma_i(E_Q) \right)^{\frac{1}{p_i}} \right]. \end{aligned}$$

Denote

$$L_Q = \frac{v_{\vec{w}}(Q) \prod_{i=1}^m \sigma_i(200B(Q))}{\mu(\alpha B(Q))^m v_{\vec{w}}(E_Q)^{\frac{1}{p}} \prod_{i=1}^m \sigma_i(E_Q)^{\frac{1}{p_i}}}$$

Now using the facts that,

$$\mu(Q)^{m(p-1)} \lesssim \mu(E_Q)^{m(p-1)} \lesssim v_{\vec{w}}(E_Q)^{\frac{p-1}{p}} \prod_{i=1}^m \sigma_i(E_Q)^{\frac{p-1}{p_i}}$$

As $p \geq \max_i p'_i$ and $E_Q \subset Q$, we have $\sigma_i(E_Q)^{\frac{p}{p_i}} \leq \sigma_i(200B(Q))^{\frac{p}{p_i}-1} \sigma_i(E_Q)$ for all $i = 1, \dots, m$. Hence,

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{A}_S(\vec{f}_\sigma) g v_{\vec{w}} d\mu &\lesssim (\sup_Q L_Q) \|M_{v_{\vec{w}} d\mu}^{\mathcal{D}}(g)\|_{L^{p'}(v_{\vec{w}} d\mu)} \prod_{i=1}^m \|M_{\sigma_i d\mu}(f_i)\|_{L^{p_i}(\sigma_i)} \\ &\lesssim (\sup_Q L_Q) \|g\|_{L^{p'}(v_{\vec{w}} d\mu)} \prod_{i=1}^m \|f_i\|_{L^{p_i}(\sigma_i)}. \end{aligned}$$

Similar to the previous case if we restrict ourselves to doubling measures one can see that $C_\omega = (\sup_Q L_Q) \leq [\vec{w}]_{A_{\vec{P}}}$. Therefore, by duality we get the result.

Finally, we consider the general case of exponents. Note that the operator \mathcal{A}_S is a multi-linear self-adjoint operator with respect to each j , $1 \leq j \leq m$. Hence without loss of generality we may assume that $p'_1 \geq \max(p, p'_2, \dots, p'_m)$. The self-adjointness of the operator together with the previous estimate and Lemma 3.6 yields

$$\begin{aligned} \|\mathcal{A}_S\|_{L^{p_1}(w_1) \times \dots \times L^{p_m}(w_m) \rightarrow L^p(v_{\vec{w}})} &= \|\mathcal{A}_S\|_{L^{p'}(v_{\vec{w}}^{1-p'}) \times \dots \times L^{p_m}(w_m) \rightarrow L^{p'_1}(w_1^{1-p'_1})} \\ &\lesssim [\vec{w}^1]_{A_{\vec{P}^1}} \\ &= [\vec{w}]_{A_{\vec{P}}}^{\frac{p'_1}{p}}. \end{aligned}$$

This completes the proof of Proposition 2.3.

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